Scientific Seminar In Applied Mathematics
Variational Approach To Shape Derivatives

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Winter Term 2011

Contents

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1 General Setting and Preliminaries

This seminar paper is intended to work out in detail the concept of shape derivatives, which is presented in the paper of [IKPa]. The ideas and results discussed in the following are mainly based on this work.

The main task of this paper is to investigate the shape differentiability of the functional

$$J(u,\Omega,\Gamma) := \int_\Omega j_1(u)dx + \int_\Gamma j_2(u)ds + \int_{\partial\Omega\setminus\Gamma} j_3(u)ds \tag{1}$$

subject to

$$E(u,\Omega) = 0. \tag{2}$$

Here, we impose the following assumptions on the cost functional J and the constraint E. For this case, let $U \subset \mathbb{R}^d$ be a fixed bounded domain with $C^{1,1}$-boundary ∂U and $D \subset U$ be a domain such that $\bar{D} \subset U$. Moreover, D is supposed to have a $C^{1,1}$-boundary $\Gamma := \partial D$ too. For the domain $\Omega \subset U$, the following three cases are admissible

(a) $\Omega = D$,
(b) $\Omega = U$,
(c) $\Omega = U\setminus\bar{D}$.

Furthermore, $E(u,\Omega)$ stands for the weak formulation of a partial differential equation stated on the domain Ω, which defines the state $u : \Omega \to \mathbb{R}^l$, $l \in \mathbb{N}_{\geq 1}$, of the considered system.

Nevertheless, one could also work with a domain $U \subset \mathbb{R}^d$, which is convex but has only a Lipschitzean boundary ∂U. We will see in the following that it is not essential to restrict oneself to a certain type of the domain U; however, it will be cleary mentionned if the additional $C^{1,1}$-regularity is necessary at some point. Besides, it is also possible to use a different constraint $E(u,\Omega)$ instead of a partial differential equation, e.g. an integral equation or an explicit formula for the state u. At the beginning of the next section, we will axiomatize the properties of the admissible constraints.

Remark 1.1. In the situation of case (a), we have $\partial\Omega = \partial D = \Gamma$, in case (b), $\partial\Omega = \partial U$ and for case (c), one finds $\partial\Omega = \partial(\mathcal{C}\Omega) = \partial(\mathcal{C}U \cup \bar{D}) = \partial(\mathcal{C}U) \cup \partial\bar{D} = \partial U \cup \partial D = \Gamma \cup \partial U$.

In order to study the shape differentiability of J, we are interested in the dependence of J on the domain Ω. Hence, we define certain mappings, which "slightly deform" the domain Ω into a different domain, and compare the values of J for these two different domains. The mappings, which are used here to realize such "deformations", are so called *perturbations of identity.*

Definition 1.2. For $h \in C^{1,1}(\bar{U},\mathbb{R}^d)$, $h|_{\partial U} = 0$ and $t \in \mathbb{R}$ define $F_t : U \to \mathbb{R}^d$ via $F_t := id + th$.

Proposition 1.3. *There exists a $\tau > 0$ such that $F_t(U) = U$ and F_t is a C^1-diffeomorphism on U for all $t \in \mathbb{R}$ with $|t| < \tau$.*

Proof. 1. $F_t(U) \subset U$: Let $x \in U$, then $\text{dist}(x,\partial U) > 0$. Since ∂U is compact, there exists a $x_0 \in \partial U$ such that $\|x - x_0\|_2 = \text{dist}(x,\partial U)$. Using an estimate for vector-valued functions related to the mean-value-theorem for scalar-valued functions we find

$$\|F_t(x) - x\|_2 = |t|\|h(x)\|_2 = |t|\|h(x) - h(x_0)\|_2 \leq |t|M_1\|x - x_0\|_2 < \|x - x_0\|_2 = \text{dist}(x,\partial U)$$

if $|t| < M_1^{-1}$. Thus, $F_t(x) \in U$ for all $t \in \mathbb{R}$ satisfying $|t| < M_1^{-1}$.

2. DF_t is regular on U: At first, we show that $\|Dh\|$ is bounded on U. Choose $x_0 \in U$ arbitrarily, then, due to the boundedness of U and the Lipschitz-continuity of Dh, there exist constants $K, L > 0$ such that for all $x \in U$, $\|Dh(x)\| \leq \|Dh(x) - Dh(x_0)\| + \|Dh(x_0)\| \leq L\|x - x_0\| + \|Dh(x_0)\| \leq KL + \|Dh(x_0)\|$, which shows this first claim. Now, let $x \in U$. Then there exists a constant $C \in (0,1)$ such that

$$\|I - DF_t(x)\| = |t|\|Dh(x)\| \leq C < 1$$

for all $t \in \mathbb{R}$ with $|t| < M_2^{-1}$, where $M_2 := \sup_{x\in U}\|Dh(x)\| + 1$. Thus, DF_t is regular for all $x \in U$ if

$$|t| < \tau := \min(M_1^{-1}, M_2^{-1}). \tag{3}$$

3. $\underline{F_t \text{ is injective}}$: Let $t \in \mathbb{R}$, $|t| < \tau$, $x, y \in U$ and $F_t(x) = F_t(y)$, then $\|x - y\| = |t|\|h(x) - h(y)\|$. If $[x, y] \subset U$, then we find $\|h(x) - h(y)\| \leq M_1\|x - y\|$ analogously to the first part of the proof. However, we can show that the claim also holds if $[x, y] \not\subseteq U$. In this case, we consider $z(s) := (1 - s)x + sy$, $s_x := \min\{s \in (0, 1)|z(s) \in \partial U\}$, $s_y := \max\{s \in (0, 1)|z(s) \in \partial U\}$, $x_0 := z(s_x)$, $y_0 := z(s_y)$; these terms are well-defined since ∂U is at least Lipschitzean. Using that $h = 0$ on ∂U, one finds $\|h(x)-h(y)\| \leq \|h(x)-h(x_0)\|+\|h(y)-h(y_0)\| \leq M_1(\|x-x_0\|+\|y-y_0\|) \leq M_1\|x-y\|$. Therefore, if $x \neq y$,

$$\|x - y\| = |t|\|h(x) - h(y)\| \leq |t|M_1\|x - y\| < \|x - y\|,$$

a contradiction. Thus, $x = y$, and F_t is injective.

4. $\underline{F_t : U \to F_t(U) \text{ is a } C^1\text{-diffeomorphism}}$: We use a result known from the lecture Analysis 2, the local invertibility:

   Let $U \subset \mathbb{R}^d$ be open, $f \in C^1(U, \mathbb{R}^d)$ and $Df(x_0)$ be regular at $x_0 \in U$. Then there exists an open set $O \subset \mathbb{R}^d$ with $x_0 \in O$ such that

   - *$f|_O$ is injective,*
   - *$f(O)$ is open,*
   - *$g := (f|_O)^{-1} \in C^1(f(O), \mathbb{R}^d)$ and $Dg(y) = (Df(g(y)))^{-1}$.*

   Here, we apply this to each $x \in U$ and notice that for $|t| < \tau$ due to the global injectivity of F_t all the local invers functions g_x coincide with F_t^{-1} on each set $F_t(O_x)$. And since $F_t(U) = F_t(\cup_{x\in U} O_x) = \cup_{x\in U} F_t(O_x)$, we know that F_t^{-1} is continuously differentiable on the open set $F_t(U)$, which proofs the claim.

5. $\underline{F_t(U) = U}$: Let $|t| < \tau$. We know that U is connected and $F_t(U) \neq \emptyset$ since $U \neq \emptyset$. Moreover, $F_t(U)$ is open as we have already seen, and $F_t(\bar{U})$ is compact since $\bar{U}$ is compact and F_t is continuous. Furthermore, $F_t(\bar{U}) = F_t(U \cup \partial U) = F_t(U) \cup F_t(\partial U)$ yields

$$F_t(\bar{U}) \cap U = (F_t(U) \cap U) \cup (F_t(\partial U) \cap U) = F_t(U) \cup (\partial U \cap U) = F_t(U)$$

and, therefore, $F_t(U)$ is relatively open and relatively closed in U. Hence, $F_t(U) = U$ and the proof is complete. □

Definition 1.4. For $t \in \mathbb{R}$ define $\Omega_t := F_t(\Omega)$ and $\Gamma_t := F_t(\Gamma)$, the perturbed domains and manifolds.

Proposition 1.5. *$\Omega_t \subset U$, $\Gamma_t \subset U$, $\partial\Omega_t \backslash \Gamma_t = \partial\Omega \backslash \Gamma$ and Γ_t is of class $C^{1,1}$ for all $t \in \mathbb{R}$ with $|t| < \tau$.*

Proof. Let $t \in \mathbb{R}$, $|t| < \tau$, then $\Omega_t = F_t(\Omega) \subset F_t(U) \subset U$ and $\Gamma_t = F_t(\Gamma) \subset F_t(U) \subset U$. If $\Omega = D$, we have $\partial\Omega_t\backslash\Gamma_t = \Gamma_t\backslash\Gamma_t = \Gamma\backslash\Gamma = \partial\Omega\backslash\Gamma$; if $\Omega = U$, we find $\partial\Omega_t\backslash\Gamma_t = \partial U\backslash\Gamma_t = \partial U = \partial\Omega\backslash\Gamma$; and if $\Omega = U\backslash\bar{D}$, one finds $\partial\Omega_t\backslash\Gamma_t = (\Gamma_t \cup \partial U)\backslash\Gamma_t = \partial U = \partial\Omega\backslash\Gamma$.

Since $F_t \in C^{1,1}(U, \mathbb{R}^d)$ and Γ is of class $C^{1,1}$, one can deduce that Γ_t is $C^{1,1}$ since the $C^{1,1}$-parametrizations of Γ, γ_i for $i \in \{1, ..., m\}$ with $m \in \mathbb{N}_{\geq 1}$, may be transfered to $C^{1,1}$-parametrizations of Γ_t, $F_t \circ \gamma_i$ for $i \in \{1, ..., m\}$. □

Definition 1.6.
1. Let $t \in \mathbb{R}$, $|t| < \tau$. If $E(u, \Omega_t) = 0$ admits a unique solution, then we denote it by $u_t : \Omega_t \to \mathbb{R}^l$ and define $u^t : \Omega \to \mathbb{R}^l$ via $u^t := u_t \circ F_t$.

2. For $t \in \mathbb{R}$, $|t| < \tau$ and $u : \Omega \to \mathbb{R}^l$ we define

$$\tilde{E}(u, t) := E(u \circ F_t^{-1}, \Omega_t),$$

   provided the right-hand-side is well-defined.

3. The *Eulerian derivative* of J at Ω in the direction $h \in C^{1,1}(\bar{U}, \mathbb{R}^d)$ is defined via

$$dJ(u, \Omega, \Gamma)h := \lim_{t\to 0} \frac{1}{t}(J(u_t, \Omega_t, \Gamma_t) - J(u, \Omega, \Gamma)).$$

4. J is called *shape differentiable* at Ω if $dJ(u,\Omega,\Gamma)h$ exists for all $h \in C^{1,1}(\bar{U},\mathbb{R}^d)$ and $dJ(u,\Omega,\Gamma) \in \mathcal{L}(C^{1,1}(\bar{U},\mathbb{R}^d),\mathbb{R})$.

The main advantage of $\tilde{E}$ in comparison to E is the fact that the solution space of

$$E(u_t,\Omega_t) = 0 \tag{4}$$

is different for each $t \in \mathbb{R}$, since $E(u_t,\Omega_t) = 0$ is a PDE posed on the domain Ω_t. However,

$$\tilde{E}(u^t,t) = 0 \tag{5}$$

is a PDE, which is equivalent to $E(u_t,\Omega_t) = 0$ in the sense that the solutions are related by $u^t = u_t \circ F_t$, respectively $u_t = u^t \circ F_t^{-1}$. But in contrast to $E(u_t,\Omega_t) = 0$, the solution space of $\tilde{E}(u^t,t) = 0$ is the same for all (sufficiently small) $t \in \mathbb{R}$, since $\tilde{E}(u^t,t) = 0$ is a PDE defined on the reference domain Ω.

Remark 1.7. From the definitions above, we conclude the following. Let $u : \Omega \to \mathbb{R}^l$ be a function, for which $E(u,\Omega)$ is well-defined. Then $\tilde{E}(u,0) = E(u \circ F_0^{-1},\Omega_0) = E(u,\Omega)$.

2 Axiomatic Description and Auxiliary Results

Consider the shape functional $J(u,\Omega,\Gamma)$ together with the constraint $E(u,\Omega)$ as stated in (1) and (2). In addition to the assumptions made at the beginning of the first section concerning U, D, Ω and Γ, we now state some further assumptions on the functions E and $\tilde{E}$ and the functionals j_1, j_2 and j_3.

(H1) There exists a Hilbert space X and a function $\tilde{E} \in C^1(X \times (-\tau,\tau), X^*)$ such that

- $E(u_t,\Omega_t) = 0$ is equivalent to $\tilde{E}(u^t,t) = 0$ in X^*,
- $\tilde{E}(u,0) = E(u,\Omega)$ for all $u \in X$.

(H2) There exists a $\tau_0 \in \mathbb{R}$, $0 < \tau_0 < \tau$ such that for all $t \in \mathbb{R}$, $|t| < \tau_0$ there exists a unique solution $u^t \in X$ of $\tilde{E}(u^t,t) = 0$. Furthermore, these solutions satisfy

$$\lim_{t\to 0} \frac{\|u^t - u^0\|_X}{t^{1/2}} = 0.$$

(H3) $E_u(u,\Omega) \in \mathcal{L}(X,X^*)$ satisfies

$$\langle E(v,\Omega) - E(u,\Omega) - E_u(u,\Omega)(v-u), \psi\rangle_{X^*\times X} = O(\|v-u\|_X^2)$$

for all $\psi \in X$ together with $u, v \in X$.

(H4) E and $\tilde{E}$ satisfy

$$\lim_{t\to 0} \frac{1}{t} \langle (\tilde{E}(u^t,t) - \tilde{E}(u,t)) - (E(u^t,\Omega) - E(u,\Omega)), \psi\rangle_{X^*\times X} = 0$$

for all $\psi \in X$ together with the solutions of (2) and (5), u and u^t.

(H5) $j_i \in C^{1,1}(\mathbb{R}^l,\mathbb{R})$ for all $i \in \{1,2,3\}$.

The first assumption (H1) requires a certain smoothness of the partial differential equation, which is assumed to appear in the weak formulation. In this process a function $u \in X$ and a parameter $t \in (-\tau,\tau)$ are mapped to a functional in X^* which can be tested with all functions $\psi \in X$. The two requirements below correspond to the idea of transforming the constraint on Ω_t into one on Ω, analogously to Definition 1.6 and Remark 1.7.

With (H2) a typical assumption on a unique solution of $\tilde{E}(u^t,t) = 0$ for sufficiently small $t \in (-\tau,\tau)$ is made. Moreover, the solutions u^t should not only converge to u^0 in the X-norm but even "faster" than $t^{1/2}$.

In (H3) a differentiability assumption on $E(u,\Omega)$ as a function of u is stated. In this case the possibility of a Taylor-expansion of $E(\cdot,\Omega)$ up to first order is required. Besides, this condition cannot be weakened to $o(\|v-u\|_X)$ since we will need in the main theorem that the left-hand-side of the equation divided through $\|v-u\|_X^2$ is bounded for $\|v-u\|_X \to 0$.

(H4) requires that the difference of the two differences, tested with every $\psi \in X$, in the brackets converges even "faster" to zero than t. This is mainly a technical assumption which will be used essentially in the main theorem in the next chapter.

Finally, (H5) states a continuity assumption on all the single functionals appearing in $J(u,\Omega,\Gamma)$. Nevertheless, this assumption cannot be weakened anymore since we will make use of the Lipschitz-continuity of j_i', $i \in \{1,2,3\}$, throughout the proofs to come.

Remark 2.1. Further, we assume $X \hookrightarrow L^2(\Omega,\mathbb{R}^l)$ and that every $x \in X$ admits a trace in $L^2(\Gamma,\mathbb{R}^l)$ or in $L^2(\partial\Omega\backslash\Gamma,\mathbb{R}^l)$ if $j_2 \neq 0$ or $j_3 \neq 0$.

Lemma 2.2. *1. The cost functional $J(u,\Omega,\Gamma)$ is well-defined for all $u \in X$.*

2. $E(u_t,\Omega_t) = 0$ has a unique solution u_t, which satisfies $u_t = u^t \circ F_t^{-1}$, provided $|t| < \tau_0$.

Proof. 1. Let $u \in X$ and $i \in \{1,2,3\}$. For $i \in \{2,3\}$ we denote the trace of u on Γ respectively $\partial\Omega\backslash\Gamma$ again with u, then u is an L^2-function on the corresponding set of definition. Due to (H5), j_i' is Lipschitz-continuous with Lipschitz-constant $L > 0$, hence, there exists a constant $C > 0$ such that for all $y \in \mathbb{R}^l$ the following estimate holds; $\xi(y)$ denotes a point which is a convex combination of y and 0.

$$\begin{aligned}|j_i(y)| &\leq |j_i(y) - j_i(0)| + |j_i(0)| = |j_i'(\xi(y))(y-0)| + |j_i(0)| \leq \|j_i'(\xi(y))\|\|y\| + |j_i(0)| \\ &\leq (\|j_i'(\xi(y)) - j_i'(0)\| + \|j_i'(0)\|)\|y\| + |j_i(0)| \leq (L\|\xi(y) - 0\| + \|j_i'(0)\|)\|y\| + |j_i(0)| \\ &\leq C(\|y\|^2 + \|y\| + 1).\end{aligned}$$

Therefore, with $x \in \mathbb{R}^l$, one finds

$$|j_i(u(x))| \leq C(\|u(x)\|^2 + \|u(x)\| + 1)$$

and deduces that $j_i(u(x))$ is integrable since u is an L^2-function and the corresponding set of definition is bounded. Thus, the cost functional is well-defined.

2. At first, (H2) implies that $\tilde{E}(u^t,t) = 0$ has a unique solutnion u^t for $|t| < \tau_0$; then, (H1) yields that $E(u_t,\Omega_t) = 0$ has a unique solutnion u_t for $|t| < \tau_0$. The identity $u_t = u^t \circ F_t^{-1}$ is a direct result of the definition of u^t.

□

Lemma 2.3. *There exists a constant $C > 0$ such that for all $i \in \{1,2,3\}$ and $u,v \in X$,*

$$\|j_i(v) - j_i(u) - j_i'(u)(v-u)\|_{L^1} \leq C\|v-u\|_X^2.$$

Proof. Due to the Mean-Value-Theorem, there exists a $\xi(x)$ for every $x \in \Omega$ such that $j_1(v(x)) - j_1(u(x)) = j_1'(\xi(x))(v(x) - u(x))$. If $L > 0$ denotes the Lipschitz-constant of j_1', we find

$$\begin{aligned}\|j_1(v) - j_1(u) - j_1'(u)(v-u)\|_{L^1} &= \int_\Omega |j_1(v) - j_1(u) - j_1'(u)(v-u)|dx \\ &= \int_\Omega |(j_1'(\xi) - j_1'(u))(v-u)|dx \leq \int_\Omega L|v-u|^2 dx \leq C\|v-u\|_X^2.\end{aligned}$$

Analogously, one proofs the claim for j_2 and j_3, where Ω has to be replaced by Γ respectively $\partial\Omega\backslash\Gamma$. □

Notation 2.4. In the sequel, n denotes the outer unit normal vector to Ω. Moreover, we use

- $\mathcal{T} := [-\tau_0, \tau_0]$,
- $I_t := \det DF_t$,
- $A_t := (DF_t)^{-T}$,
- $w_t := I_t\|A_t n\|$.

Remark 2.5. For the following proposition, we need the concept of the *surface divergence* of a function $\varphi \in C^1(\bar{U}, \mathbb{R}^d)$, which is defined as

$$\operatorname{div}_M \varphi := \operatorname{div} \varphi|_M - (D\varphi n) \cdot n$$

where $M \subset \bar{U}$ is a submanifold of dimension $d-1$ and n denotes the unit normal vector to M.

This definition is similar to that of the intrinsic derivative [Fra97] of a vector-valued function V defined on a curve $C \subset M$, where M is supposed to be a two-dimensional submanifold of $\mathbb{R}^3$. If C is parametrised by t, one defines

$$\frac{\nabla V}{dt} := \frac{dV}{dt} - \left(\frac{dV}{dt}, n\right) n = \sum_{i=1}^{3} \left(\frac{dV}{dt}, e_i\right) e_i - \left(\frac{dV}{dt}, n\right) n$$

together with $\{e_i\}_{1 \le i \le 3}$, the canonical basis of $\mathbb{R}^3$. Here, the normal component of dV/dt is subtracted, hence, $\nabla V/dt$ is the projection of dV/dt onto M.

In a similar way, the surface divergence is defined. If $\{e_i\}_{1 \le i \le d}$ is the canonical basis of $\mathbb{R}^d$, then

$$\operatorname{div}_\Gamma \varphi = \operatorname{tr} D\varphi - (D\phi n) \cdot n = \sum_{i=1}^{d} (D\varphi e_i) \cdot e_i - (D\varphi n) \cdot n.$$

Thus, it is consistent to define the surface divergence in the form described above.

Proposition 2.6. *Let $h \in C^{1,1}(\bar{U}, \mathbb{R}^d)$ and $F_t = id + th$ for $t \in \mathcal{T}$. Then*

$$\begin{aligned}
&t \mapsto F_t \in C^1(\mathcal{T}, C^{1,1}(\bar{U}, \mathbb{R}^d)), && t \mapsto F_t^{-1} \in C(\mathcal{T}, C^1(\bar{U}, \mathbb{R}^d)),\\
&t \mapsto I_t \in C^1(\mathcal{T}, C(\bar{U})), && t \mapsto A_t \in C(\mathcal{T}, C(\bar{U}, \mathbb{R}^{d \times d})),\\
&t \mapsto w_t \in C(\mathcal{T}, C(\Gamma)), && \\
&\tfrac{d}{dt} F_t|_{t=0} = h, && \tfrac{d}{dt} F_t^{-1}|_{t=0} = -h,\\
&\tfrac{d}{dt} DF_t|_{t=0} = Dh, && \tfrac{d}{dt} (DF_t)^{-1}|_{t=0} = \tfrac{d}{dt} A_t^T|_{t=0} = -Dh,\\
&\tfrac{d}{dt} I_t|_{t=0} = \operatorname{div} h, && \tfrac{d}{dt} w_t|_{t=0} = \operatorname{div}_\Gamma h.
\end{aligned}$$

Proof. Throughout the proof, we use the notation $t_n \to t_*$ for a sequence $(t_n)_{n \in \mathbb{N}} \subset \mathcal{T}$ which converges to a given $t_* \in \mathcal{T}$.

$t \mapsto F_t \in C^1(\mathcal{T}, C^{1,1}(\bar{U}, \mathbb{R}^d))$: $t_n \to t_* \;\Rightarrow\; \|F_{t_n} - F_{t_*}\|_{C^{1,1}} = |t_n - t_*| \|h\|_{C^{1,1}} \to 0$ and

$$\frac{d}{dt} F_t|_{t=t_0} = \lim_{\Delta t \to 0} \frac{(t_0 + \Delta t)h - t_0 h}{\Delta t} = h,$$

which is constant and, therefore, also continous in t with respect to the $C^{1,1}$-norm.

$t \mapsto F_t^{-1} \in C(\mathcal{T}, C^1(\bar{U}, \mathbb{R}^d))$: $F_t^{-1} \in C^1(U, \mathbb{R}^d)$ since F_t is a diffeomorphism on U. Moreover, $DF_t(y)$ is regular for all $y \in \partial U$, if one considers the continuous extension of DF_t to $\bar{U}$. This is true because due to (3) we know that $\|I - DF_t(y')\| \le C < 1$ for all $y' \in U$ and therefore $\|I - DF_t(y)\| \le C < 1$ too. Thus, we may for $y \in \partial U$ define $F_t^{-1}(y) := y$ and $D(F_t^{-1})(y) := (DF_t(y))^{-1}$, a candidate for the C^1-extension of F_t^{-1}. Let $t \in \mathcal{T}$ be fixed, $y_n \to y_*$ with $(y_n)_n \subset U$, $y_* \in \partial U$. Assume

$$F_t^{-1}(y_n) =: x_n \not\to y_* = F_t^{-1}(y_*),$$

then $(x_n)_n$ has a subsequence $(x_{n_k})_k \to x' \in \bar{U} \backslash \{y_*\}$. If $x' \in U$, then $y_{n_k} := F_t(x_{n_k}) \to F_t(x') \in U$, a contradiction; and if $x' \in \partial U \backslash \{y_*\}$, then $y_{n_k} \to F_t(x') = x'$, a contradiction, hence, $F_t^{-1} \in C(\bar{U}, \mathbb{R}^d)$. Using this, we find

$$y_n \to y_* \Rightarrow F_t^{-1}(y_n) \to y_* \Rightarrow DF_t(F_t^{-1}(y_n)) \to DF_t(y_*) \Rightarrow D(F_t^{-1})(y_n) \to D(F_t^{-1})(y_*)$$

as a result of the definition made above; consequently, $F_t^{-1} \in C^1(\bar{U}, \mathbb{R}^d)$.

It remains to show that

$$t_n \to t_* \Rightarrow F_{t_n}^{-1} \to F_{t_*}^{-1} \text{ in } \; C^1.$$

For this case, assume first that $F_{t_n}^{-1} \nrightarrow F_{t_*}^{-1}$ in C^0, i.e.

$$\exists \epsilon > 0 \; \forall N \in \mathbb{N} \; \exists n_N \geq N, \; y_N \in \bar{U} \;\; : \;\; \|F_{t_{n_N}}^{-1}(y_N) - F_{t_*}^{-1}(y_N)\| \geq \epsilon.$$

Then there exists a sequence $(N_k)_k$ such that

$$y_{n_k} \to y, \quad x_{N_k} := F_{t_{n_{N_k}}}^{-1}(y_{N_k}) \to x \in \bar{U} \quad \text{and} \quad x_{N_k,*} := F_{t_*}^{-1}(y_{N_k}) \to x_* \in \bar{U}.$$

In addition, we know that $F_{t_n}(x_n) \to F_t(x)$ for $t_n \to t$ and $x_n \to x$. All together, we find $\|x - x_*\| \geq \epsilon > 0$ due to the assumption above and that

$$\begin{aligned} y &= \lim_{k\to\infty} y_{N_k} = \lim_{k\to\infty} F_{t_{n_{N_k}}}(x_{N_k}) = F_{t_*}(x) \quad \text{and} \\ y &= \lim_{k\to\infty} y_{N_k} = \lim_{k\to\infty} F_{t_*}(x_{N_k,*}) = F_{t_*}(x_*), \end{aligned}$$

a contradiction to the injectivity of F_{t_*}. In order to show that $D(F_{t_n}^{-1}) \to D(F_{t_*}^{-1})$ in C^0, we observe that $F_{t_n}^{-1}(y) \to F_{t_*}^{-1}(y)$ uniformly in y. Again using that $DF_{t_n}(x_n) \to DF_t(x)$ for $t_n \to t$ and $x_n \to x$, one finds that $DF_{t_n}(F_{t_n}^{-1}(y)) \to DF_{t_*}(F_{t_*}^{-1}(y))$ uniformly in y and hence

$$(DF_{t_n}(F_{t_n}^{-1}(y)))^{-1} \to (DF_{t_*}(F_{t_*}^{-1}(y)))^{-1} \quad \text{uniformly in } y.$$

<u>$t \mapsto I_t \in C^1(\mathcal{T}, C(\bar{U}))$:</u> We first need to show that $\det : \mathbb{R}^{d\times d} \to \mathbb{R}$ is a C^1-function. But since for a matrix $A = (a_{ij})_{ij} \in \mathbb{R}^{d\times d}$ the representation

$$\det A = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sign}(\sigma) \prod_{k=1}^{d} a_{\sigma(k),k}$$

holds, which consists of sums and products of functions which themselves depend continuously differentiable on A, the claim follows and $I_t \in C(\bar{U})$.

Now, let $t_n \to t_*$, then $\|DF_{t_n} - DF_{t_*}\|_{C^0} = |t_n - t_*|\|Dh\|_{C^0} \to 0$ and, consequently, $I_{t_n} \to I_{t_*}$ in C^0. In addition,

$$\frac{d}{dt} I_t|_{t=t_0} = (D\det(DF_{t_0}))Dh \quad \text{where} \quad D\det(DF_{t_0}) \in \mathcal{L}(\mathbb{R}^{d\times d}, \mathbb{R}) \cong \mathbb{R}^{d\times d},$$

however, the application of $D\det(DF_{t_0})$ to Dh is not given by the usual matrix multiplication. Nevertheless, we have

$$\begin{aligned} \left\| \frac{d}{dt} I_t|_{t=t_n} - \frac{d}{dt} I_t|_{t=t_*} \right\|_{C^0} &= \max_{x\in\bar{U}} |(D\det(DF_{t_n}))Dh(x) - (D\det(DF_{t_*}))Dh(x)| \\ &\leq \max_{x\in\bar{U}} \|D\det(DF_{t_n}) - D\det(DF_{t_*})\|_{\mathcal{L}(\mathbb{R}^{d\times d},\mathbb{R})} \|Dh(x)\| \\ &\to 0 \end{aligned}$$

since $\det \in C^1(\mathbb{R}^{d\times d}, \mathbb{R})$. This proves the claim.

<u>$t \mapsto A_t \in C(\mathcal{T}, C(\bar{U}, \mathbb{R}^{d\times d}))$:</u> $A_t \in C(\bar{U}, \mathbb{R}^{d\times d})$ since DF_t, inverting and transposing of matrices are continuous functions. For $t_n \to t_*$, we know that $DF_{t_n} \to DF_{t_*}$ in C^0 and therefore $A_{t_n} \to A_{t_*}$ in C^0 as desired.

<u>$t \mapsto w_t \in C(\mathcal{T}, C(\Gamma))$:</u> Since Γ is of class $C^{1,1}$, the vector n continuously depends on $x \in \Gamma$, and due to the continuity of A_t and I_t, we have $w_t \in C(\Gamma)$. Now, let $t_n \to t_*$, then

$$\begin{aligned} \|w_{t_n} - w_{t_*}\|_{C(\Gamma)} &= \|I_{t_n}\|A_{t_n}n\| - I_{t_*}\|A_{t_*}n\|\|_{C(\Gamma)} \\ &\leq \|(I_{t_n} - I_{t_*})\|A_{t_n}n\|\|_{C(\Gamma)} + \|I_{t_*}(\|A_{t_n}n\| - \|A_{t_*}n\|)\|_{C(\Gamma)} \\ &\leq \|I_{t_n} - I_{t_*}\|_{C(\Gamma)}\|A_{t_n}n\|_{C(\Gamma,\mathbb{R}^{d\times d})} + \|I_{t_*}\|_{C(\Gamma)}\|(A_{t_n} - A_{t_*})n\|_{C(\Gamma,\mathbb{R}^{d\times d})} \\ &\to 0 \end{aligned}$$

due to the properties of I_t and A_t.

Calculation of the derivatives: Some of the derivatives can be easily calculated using also the derivative of the matrix inversion $\mathrm{inv}: GL_n(\mathbb{R}) \to GL_n(\mathbb{R}),\ A \mapsto A^{-1}$ for which the directional derivative is given by $D\,\mathrm{inv}(A)\delta A = -A^{-1}\delta A A^{-1}$ for all $\delta A \in \mathbb{R}^{d\times d}$.

$$\begin{aligned}
\frac{d}{dt}F_t|_{t=0} &= \lim_{\Delta t\to 0}\frac{id + h\Delta t - id}{\Delta t} = h,\\
\frac{d}{dt}F_t^{-1}|_{t=0} &= \lim_{\Delta t\to 0}\frac{F_{\Delta t}^{-1} - F_0^{-1}}{\Delta t} = \lim_{\Delta t\to 0}\frac{(id + h\Delta t + o_1(\Delta t))^{-1} - id}{\Delta t}\\
&= \lim_{\Delta t\to 0}\frac{id - h\Delta t - o_1(\Delta t) + o_2(h\Delta t + o_1(\Delta t)) - id}{\Delta t}\\
&= -h + \lim_{\Delta t\to 0}\frac{o_2(h\Delta t + o_1(\Delta t))}{\|h\Delta t + o_1(\Delta t)\|}\frac{\|h\Delta t + o_1(\Delta t)\|}{\Delta t} = -h,\\
\frac{d}{dt}DF_t|_{t=0} &= \lim_{\Delta t\to 0}\frac{I + Dh\Delta t - I}{\Delta t} = Dh,\\
\frac{d}{dt}A_t^T|_{t=0} &= \frac{d}{dt}(DF_t)^{-1}|_{t=0} = -I^{-1}\left(\frac{d}{dt}(DF_t)|_{t=0}\right) I = -Dh.
\end{aligned}$$

For a matrix $A \in \mathbb{R}^{d\times d}$, we have $D\det(A) \in \mathcal{L}(\mathbb{R}^{d\times d}, \mathbb{R})$, hence, $D\det(A)$ can be considered as a matrix again. But the question is, what are the entries in this matrix and how can the application of $D\det(A)$ to a matrix $h \in \mathbb{R}^{d\times d}$ be described? The result may be intuitive, but nevertheless, we will prove it in detail. For this, we claim that for a function $f \in C^1(\mathbb{R}^{d\times d}, \mathbb{R})$ and matrices $A, h \in \mathbb{R}^{d\times d}$ the following holds:

$$Df(A) = \left(\frac{\partial}{\partial X_{ij}} f(X)|_{X=A}\right)_{ij} \quad \text{and} \quad Df(A)h = \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial}{\partial X_{ij}} f(X)|_{X=A} h_{ij}. \tag{6}$$

Obviously, $Df(A)h$ defines a bounded linear operator on $\mathbb{R}^{d\times d}$ since f is continuously differentiable. To verify that this bounded linear operator is also the derivative of f at A, we show that

$$\lim_{h\to 0}\frac{1}{\|h\|}\left(f(A+h) - f(A) - \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial}{\partial X_{ij}} f(X)|_{X=A} h_{ij}\right) = 0.$$

We define the following matrices

$$h^{ij} := (h'_{kl})_{kl} \quad \text{where} \quad h'_{kl} := \begin{cases} h_{kl} & \text{if } k < i \text{ or } (k = i \text{ and } l \le j)\\ 0 & \text{otherwise}\end{cases}$$

which are identical to h up to a certain index and zero afterwards. Using the notation h^{ij-} for the matrix which is equal to h^{ij} except with a further zero at the "last" nontrivial entry of h^{ij}, we find

$$\begin{aligned}
\frac{1}{\|h\|}\left(f(A+h) - f(A) - \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial}{\partial X_{ij}} f(A) h_{ij}\right) &=\\
\frac{1}{\|h\|}\left(\sum_{i=1}^{d}\sum_{j=1}^{d} f(A + h^{ij}) - f(A + h^{ij-}) - \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial}{\partial X_{ij}} f(A) h_{ij}\right) &=\\
\frac{1}{\|h\|}\left(\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial}{\partial X_{ij}} f(A + h^{ij-}) h_{ij} - \sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial}{\partial X_{ij}} f(A) h_{ij}\right) &=\\
\sum_{i=1}^{d}\sum_{j=1}^{d}\left(\frac{\partial}{\partial X_{ij}} f(A + h^{ij-}) - \frac{\partial}{\partial X_{ij}} f(A)\right)\frac{h_{ij}}{\|h\|} &\to \quad 0
\end{aligned}$$

since the second factor in every summand is bounded while the first converges to zero.

If we apply this to the C^1-function $\det : \mathbb{R}^{d\times d} \to \mathbb{R}$, we have to calculate the partial derivative of det with respect to every matrix entry. But using the Laplace-formula, one finds

$$D\det(A) = \left(\frac{\partial}{\partial X_{ij}} \det(X)|_{X=A}\right)_{ij} = \left((-1)^{i+j}\delta_{ij}\right)_{ij}$$

where δ_{ij} is the determinant of the matrix A without row i and column j. Now, the remaining derivatives can be easily calculated, but we shall use the symbol $\cdot$ to denote the sum over the componentwise product of two matrices, just in analogy to the representation in (6).

$$\begin{aligned}
\frac{d}{dt} I_t|_{t=0} &= D\det(DF_t|_{t=0}) \cdot \frac{d}{dt} DF_t|t = 0 = D\det(I) \cdot Dh = I \cdot Dh = \operatorname{div} h, \\
\frac{d}{dt} w_t|_{t=0} &= \left(\frac{d}{dt} I_t|_{t=0}\right) \|A_t n|_{t=0}\| + I_t|_{t=0} \frac{(A_t n)^T}{\|A_t n\|}|_{t=0} \left(\frac{d}{dt} A_t|_{t=0}\right) n = \\
&= \operatorname{div} h|_\Gamma + n^T(-(Dh)^T)n = \operatorname{div}_\Gamma h.
\end{aligned}$$

Finally, the proof of the proposition is finished. □

Remark 2.7. The above calculations also show that each derivative exists uniformly in x, i.e. there is no explicit dependence on the spatial variable $x \in \bar{U}$.

Corollary 2.8. *There exist constants $\alpha, \beta > 0$ such that*

$$\alpha \le I_t(x) \le \beta$$

for all $t \in \mathcal{T}$ and $x \in \bar{U}$.

Proof. $I_0(x) = 1$ for all $x \in \bar{U}$ and $I_t(x) \neq 0$ for all $x \in \bar{U}$ and $t \in \mathcal{T}$ since $DF_t(x)$ is regular for those t and x. Due to Proposition 2.6, the function I_t continuously depends on the parameter t; therefore, the claim follows from the compactness of $\bar{U}$ and $\mathcal{T}$. □

3 The Shape Derivative

Lemma 3.1. 1. *Let $\varphi_t \in L^1(\Omega_t)$, then $\varphi_t \circ F_t \in L^1(\Omega)$ and*

$$\int_{\Omega_t} \varphi_t dx_t = \int_\Omega \varphi_t \circ F_t \ \det DF_t \ dx.$$

2. *Let $\psi_t \in L^1(\Gamma_t)$, then $\psi_t \circ F_t \in L^1(\Gamma)$ and*

$$\int_{\Gamma_t} \psi_t d\Gamma_t = \int_\Gamma \psi_t \circ F_t \ \det DF_t \ \|(DF_t)^{-T} n\| \ d\Gamma.$$

Proof. 1. This is a well-known result of advanced analysis. A proof can be found in [Jos05].

2. In the following, the main steps of the proof are described which is presented in [IKPb]. We start with an arbitrary $d-1$-dimensional submanifold $M \subset \mathbb{R}^d$ and a set $U \subset M$ open in M which is parametrized via $\varphi : S \to U$ with $S \subset \mathbb{R}^{d-1}$ open. Similar to the first statement of the Lemma, a function $f : M \to \mathbb{R}$ with $\operatorname{supp} f \subset U$ is integrable over U if $f \circ \varphi \ (\det(D\varphi^T D\varphi))^{1/2}$ is integrable over S and thus
$$\int_U f(x) dM := \int_S (f \circ \varphi)(t)(\det(D\varphi(x)^T D\varphi(x)))^{1/2} dt$$
is well-defined.

   Furthermore, a technical result is necessary stating the following. Let $X := (x_1, ..., x_{d-1}) \in \mathbb{R}^{d\times d-1}$ be a matrix with linearly independent columns, then $n \in \mathbb{R}^d$ defined via

$$n_i := (-1)^{i-1} \det X(\hat{i}),$$

where $X(\hat{i}) \in \mathbb{R}^{d-1\times d-1}$ is the matrix X without row i, is orthogonal to the hyperplane spanned by $x_1, ..., x_{d-1}$; moreover,

$$\|n\| = (\det(X^T X))^{1/2}.$$

The orthogonality of n to all the vectors x_i, $i \in \{1, ..., d-1\}$ is clear, since for an $i \in \{1, ..., d-1\}$ the inner product

$$(n, x_1) = \sum_{i=1}^{d} n_i x_{1i} = \det(X, x_1) = 0.$$

This is a consequence of the Laplace-formula and the fact that x_i appears twice as a column in (X, x_i). The more difficult part is for sure the representation of the norm of n, equivalently, this reads

$$\sum_{i=1}^{d} \det(X(\hat{i})^T X(\hat{i})) = \det(X^T X).$$

For a proof of this claim see [Mun91]. We shall now use this result and focus on the special situation in the statement we want to show.

By the assumption stated at the beginning of Section 1, $\Gamma = \partial D$ is the $C^{1,1}$-boundary of a domain D satisfying $\bar{D} \subset U$. Hence, one finds sets $O_1, ..., O_n \subset \mathbb{R}^d$, which are open in $\mathbb{R}^d$ and cover Γ, together with $C^{1,1}$-diffeomorphisms $c_1, ..., c_n$, $c_i : O_i \to B(0,1)$ such that

$$c_i(D \cap O_i) = \{x \in B(0,1) \mid x_n \leq 0\} \quad \text{and} \quad c_i(\Gamma \cap O_i) = \{x \in B(0,1) \mid x_n = 0\}.$$

So, the $d-1$-dimensional manifold Γ is locally transformed into the unit sphere $S_0 := \{x' \in \mathbb{R}^{d-1} \mid \|x'\| \leq 1\}$ in $\mathbb{R}^{d-1}$ considered as a subset of the d-dimensional unit sphere. Now, we know that each c_i has a $C^{1,1}$-inverse function h_i, which we further restrict to $\{x \in B(0,1) \mid x_n = 0\}$ and denote it then with $\tilde{h}_i$. Consequently, the functions $\tilde{h}_i : S_0 \to \Gamma \cap O_i$ define local patches of Γ; thus, the compositions $F_t \circ \tilde{h}_i : S_0 \to F_t(\Gamma) \cap F_t(O_i)$ define local pathches of Γ_t.

Now, if the transformation rule is proven for any ψ_t with $\operatorname{supp} \psi_t \subset F_t(\Gamma) \cap F_t(O_i)$ and i arbitrarily, then the proof is finished, since one can choose an appropriate partition of unity and apply the result to each $F_t(\Gamma) \cap F_t(O_i)$. Consequently, we consider from now on only one of these patches and omit the index i, according to [IKPb].

We define the function $\tilde{n}$ via

$$\tilde{n} \circ h := (\det Dh)(Dh)^{-T} e_d : \{x \in B(0,1) \mid x_n = 0\} \to \mathbb{R}^d$$

and find that

$$(\tilde{n} \circ h)_i = (-1)^{n+i} \det(D_{x'} h(\hat{i}))$$

due to the representation of the inverse of a matrix. Using the auxiliary result from above, we conclude that $\tilde{n} \circ h$ is orthogonal to Γ with norm

$$(\det(D_{x'} \tilde{h}^T D_{x'} \tilde{h}))^{1/2} = \|\tilde{n} \circ h\| = |\det Dh| \left\|(Dh)^{-T} e_d\right\|. \tag{7}$$

Moreover, by definition of $\tilde{n} \circ h$, we deduce

$$\begin{aligned} D(F_t \circ h)^{-T} e_d &= ((DF_t \circ h) Dh)^{-T} e_d = \left((DF_t)^{-T} \circ h\right) (Dh)^{-T} e_d \\ &= \left((DF_t)^{-T} \circ h\right) (\det Dh)^{-1} (\tilde{n} \circ h) \\ &= (\det Dh)^{-1} \left(\left((DF_t)^{-T} n\right) \circ h\right) (\|\tilde{n}\| \circ h); \end{aligned}$$

here n denotes the exterior unit normal of $\Gamma = \partial D$. We may assume that $\tilde{n} \circ h$ is a normal vector pointing to the exterior of D, otherwise one can replace $\tilde{n} \circ h$ by $-\tilde{n} \circ h$, which yieds the same result in the above calculation. In the following, we collect these preparations and start with the definition of the surface integral; in addition, we replace $\tilde{h}$ by $F_t \circ \tilde{h}$ in equation (7) and make use of Corollary 2.8. All together, we find

$$
\begin{aligned}
\int_{\Gamma_t} \psi_t(x_t) d\Gamma_t &= \int_{S_0} \psi_t \circ (F_t \circ \tilde{h}) \left(\det\left(D_{x'}(F_t \circ \tilde{h})^T D_{x'}(F_t \circ \tilde{h})\right)\right)^{1/2} dx' \\
&= \int_{S_0} \psi_t \circ (F_t \circ \tilde{h}) |\det(D(F_t \circ h))| \left\|(D(F_t \circ h))^{-T} e_d\right\| dx' \\
&= \int_{S_0} \psi_t \circ (F_t \circ \tilde{h}) |\det((DF_t \circ h) Dh)| |\det(Dh)|^{-1} \left\|(DF_t)^{-T} n\right\| \circ h \|\tilde{n}\| \circ h dx' \\
&= \int_{S_0} \psi_t \circ (F_t \circ \tilde{h}) \det(DF_t \circ h) \left\|(DF_t)^{-T} n\right\| \circ h \|\tilde{n}\| \circ h dx' \\
&= \int_{S_0} (\psi_t \circ F_t) \circ \tilde{h} \det(DF_t) \circ h \left\|(DF_t)^{-T} n\right\| \circ h \left(\det(D_{x'} \tilde{h}^T D_{x'} \tilde{h})\right)^{1/2} dx' \\
&= \int_{\Gamma} \psi_t \circ F_t \det(DF_t) \left\|(DF_t)^{-T} n\right\| d\Gamma,
\end{aligned}
$$

which finishes the proof. □

Theorem 3.2. *Let (H1) - (H5) be true and assume that the adjoint equation*

$$\langle E_u(u, \Omega)\psi, p\rangle_{X^* \times X} - (j_1'(u), \psi)_\Omega - (j_2'(u), \psi)_\Gamma - (j_3'(u), \psi)_{\partial\Omega\setminus\Gamma} = 0, \quad \psi \in X \tag{8}$$

has a unique solution $p \in X$, with u the solution of (2).

Then the Eulerian derivative of J at Ω in the direction h, $dJ(u, \Omega, \Gamma)h$, exists and

$$dJ(u, \Omega, \Gamma)h = -\frac{d}{dt} \langle \tilde{E}(u, t), p\rangle_{X^* \times X}|_{t=0} + \int_\Omega j_1(u) \operatorname{div} h \; dx + \int_\Gamma j_2(u) \operatorname{div}_\Gamma h \; ds.$$

Proof. At first, (H2) implies that there exist unique solutions $u^t \in X$ and $u \in X$ to $\tilde{E}(u^t, t) = 0$ respectively $E(u, \Omega) = 0$ for all $t \in \mathcal{T}$. Thus, $u_t = u^t \circ F_t^{-1}$ satisfies $E(u_t, \Omega_t) = 0$. The idea is now to apply Lemma 3.1, and to add and subtract appropriate terms in a useful way. This reads

$$
\begin{aligned}
& \frac{1}{t}(J(u_t, \Omega_t, \Gamma_t) - J(u, \Omega, \Gamma)) \qquad (9) \\
= \; & \frac{1}{t} \int_\Omega (I_t j_1(u^t) - j_1(u)) dx + \int_\Gamma (w_t j_2(u^t) - j_2(u)) ds + \int_{\partial\Omega\setminus\Gamma} (j_3(u^t) - j_3(u)) ds \\
= \; & \frac{1}{t} \int_\Omega (I_t(j_1(u^t) - j_1(u) - j_1'(u)(u^t - u)) + (I_t - 1) j_1'(u)(u^t - u) + j_1'(u)(u^t - u) + (I_t - 1) j_1(u)) dx \\
+ \; & \frac{1}{t} \int_\Gamma (w_t(j_2(u^t) - j_2(u) - j_2'(u)(u^t - u)) + (w_t - 1) j_2'(u)(u^t - u) + j_2'(u)(u^t - u) + (w_t - 1) j_2(u)) ds \\
+ \; & \frac{1}{t} \int_{\partial\Omega\setminus\Gamma} ((j_3(u^t) - j_3(u) - j_3'(u)(u^t - u)) + j_3'(u)(u^t - u)) ds,
\end{aligned}
$$

where the integrals over $\partial\Omega\setminus\Gamma$ do not need a transformation since $\partial\Omega_t\setminus\Gamma_t = \partial\Omega\setminus\Gamma$ per construction. As a result of Proposition 2.6, the functions I_t and w_t are bounded for each $t \in \mathcal{T}$, hence, there exist constants $c_1, c_2 > 0$ such that $I_t(x) \leq c_1$ and $w_t \leq c_2$ for all $x \in \Omega$, $y \in \Gamma$, $t \in \mathcal{T}$. Therefore, Lemma 2.3 yields the following estimates with a generic constant $c > 0$ for each inequality independent of t.

$$
\begin{aligned}
\left|\int_\Omega I_t(j_1(u^t) - j_1(u) - j_1'(u)(u^t - u)) dx\right| &\leq c_1 \int_\Omega \left|j_1(u^t) - j_1(u) - j_1'(u)(u^t - u)\right| dx \leq c\|u^t - u\|_X^2, \\
\left|\int_\Gamma w_t(j_2(u^t) - j_2(u) - j_2'(u)(u^t - u)) dx\right| &\leq c_2 \int_\Gamma \left|j_2(u^t) - j_2(u) - j_2'(u)(u^t - u)\right| dx \leq c\|u^t - u\|_X^2, \\
\left|\int_{\partial\Omega\setminus\Gamma} (j_3(u^t) - j_3(u) - j_3'(u)(u^t - u)) dx\right| &\leq c\|u^t - u\|_X^2. \qquad (10)
\end{aligned}
$$

We further use the adjoint equation and the adjoint state p together with the fact that $\tilde{E}(u^t,t) = \tilde{E}(u,0) = 0$ in X^* to find

$$\begin{aligned} (j_1'(u), u^t - u)_\Omega &+ (j_2'(u), u^t - u)_\Gamma + (j_3'(u), u^t - u)_{\partial\Omega\setminus\Gamma} = \langle E_u(u,\Omega)(u^t - u), p\rangle_{X^*\times X} \\ &= -\langle E(u^t,\Omega) - E(u,\Omega) - E_u(u,\Omega)(u^t-u), p\rangle_{X^*\times X} \\ &- \langle \tilde{E}(u^t,t) - \tilde{E}(u,t) - E(u^t,\Omega) + E(u,\Omega), p\rangle_{X^*\times X} \\ &- \langle \tilde{E}(u,t) - \tilde{E}(u,0), p\rangle_{X^*\times X}. \end{aligned} \tag{11}$$

Now, we use the assumptions (H1) - (H4) and Proposition 2.6 to estimate the ten summands appearing at the end of equations (9). In detail, the terms one, five and nine converge to zero due to equations (10) and (H2) since

$$\lim_{t\to 0} \frac{c\|u^t - u\|_X^2}{t} = c\left(\lim_{t\to 0}\frac{\|u^t-u\|_X}{t^{1/2}}\right)^2 = 0.$$

Similarly, the terms two and six converge to zero as a result of Proposition 2.6 and (H2) as

$$\begin{aligned} \lim_{t\to 0}\frac{(I_t - 1)j_1'(u)(u^t-u)}{t} &= \operatorname{div} h\; j_1'(u)\lim_{t\to 0}(u^t - u) = 0 \quad \text{and} \\ \lim_{t\to 0}\frac{(w_t - 1)j_2'(u)(u^t-u)}{t} &= \operatorname{div}_\Gamma h\; j_2'(u)\lim_{t\to 0}(u^t - u) = 0. \end{aligned}$$

The terms four and eight establish the two integrals in the final formula because Proposition 2.6 yields

$$\begin{aligned} \lim_{t\to 0}\frac{(I_t-1)j_1(u)}{t} &= \operatorname{div} h\; j_1(u) \quad \text{and} \\ \lim_{t\to 0}\frac{(w_t-1)j_2(u)}{t} &= \operatorname{div}_\Gamma h\; j_2(u). \end{aligned}$$

Finally, the remaining terms three, seven and ten sum up to the last part of equations (11) divided through t. As a consequence of (H3),

$$\begin{aligned} \lim_{t\to 0}\frac{1}{t}\langle E(u^t,\Omega) - E(u,\Omega) - E_u(u,\Omega)(u^t-u), p\rangle_{X^*\times X} &= \\ \lim_{t\to 0}\left(\frac{1}{\|u^t-u\|_X^2}\langle E(u^t,\Omega) - E(u,\Omega) - E_u(u,\Omega)(u^t-u), p\rangle_{X^*\times X}\frac{\|u^t-u\|_X^2}{t}\right) &= 0 \end{aligned}$$

since the first part of the product is bounded and the second part converges to zero due to (H2). In addition,

$$\lim_{t\to 0}\frac{1}{t}\langle \tilde{E}(u^t,t) - \tilde{E}(u,t) - E(u^t,\Omega) + E(u,\Omega), p\rangle_{X^*\times X} = 0$$

as an immediate consequence of assumption (H4). Thus, the only nontrivial part coming from equations (11) is the negative of the derivative of $\tilde{E}$ with respect to the variable t, which results in

$$\lim_{t\to 0}\frac{1}{t}\left((j_1'(u), u^t-u)_\Omega + (j_2'(u), u^t-u)_\Gamma + (j_3'(u), u^t-u)_{\partial\Omega\setminus\Gamma}\right) = -\frac{d}{dt}\langle\tilde{E}(u,t), p\rangle_{X^*\times X}|_{t=0}.$$

This shows the desired representation of the derivative of J and completes the proof. □

Remark 3.3. A closer look at the proof of Theorem 3.2 shows that the assumption (H1) is not used in its full strength; a weaker formulation containing only the necessary parts reads as follows.

(H1') There exists a Hilbert space X and a function $\tilde{E} : X \times (-\tau,\tau) \to X^*$ such that

- With u and p the solutions of equation (2) respectively equation (8) the following holds.
  - $v \to \langle\tilde{E}(v,0), p\rangle_{X^*\times X}$ is differentiable at $v = u$,
  - $t \to \langle\tilde{E}(u,t), p\rangle_{X^*\times X}$ is differentiable at $t = 0$.
- $E(u_t,\Omega_t) = 0$ is equivalent to $\tilde{E}(u^t,t) = 0$ in X^*
- $\tilde{E}(u,0) = E(u,\Omega)$ for all $u \in X$.

We additionally introduce the following assumption.

(H6) For every $f \in X^*$, the linearized equation

$$\langle E_u(u,\Omega)p, \psi\rangle_{X^* \times X} = \langle f, \psi\rangle_{X^* \times X}, \quad \psi \in X$$

has a unique solution $p \in X$, provided equation (2) admits a unique solution u.

Proposition 3.4. *If equation (2) has a unique solution u and both (H1) and (H6) hold, then (H2) holds.*

Proof. Together with u, the unique solution of $E(u,\Omega) = \tilde{E}(u,0) = 0$, we find at first that

$$\tilde{E}_u(u,0) = E_u(u,\Omega)$$

since $\tilde{E}(v,0) = E(v,\Omega)$ for all $v \in X$. (H6) now yields that $E_u(u,\Omega)$ is bijective and, hence, $\tilde{E}_u(u,0)$ is bijective. We now have three Banach-spaces X, X^* and $\mathbb{R}$, an open subset $X \times (-\tau,\tau)$ of $X \times \mathbb{R}$, a continuously differentiable function $\tilde{E} : X \times (-\tau,\tau) \to X^*$ and an element $(u,0) \in X \times (-\tau,\tau)$ such that $\tilde{E}(u,0) = 0$ and $\tilde{E}_u(u,0) \in \mathcal{L}(X,X^*)$ is an isomorphism between X and X^*. The latter is true due to the theorem about the continuous inverse stating that the inverse of a bijective bounded linear operator between Banach spaces is also a bounded linear operator. Hence, the generalized implicit function theorem [Jos05] can be applied and one finds that there exist neighbourhoods $U \subset X$ of u and $(-\tau_0,\tau_0) \subset (-\tau,\tau)$ of 0 and a differentiable function $g : (-\tau_0,\tau_0) \to U$ such that

$$\tilde{E}(g(t),t) = 0 \text{ for all } t \in (-\tau_0,\tau_0).$$

Moreover, $g(t)$ is the only solution of $\tilde{E}(u^t,t) = 0$ in U for all $t \in (-\tau_0,\tau_0)$. Therefore, there exists a unique solution $u^t := g(t) \in U$ for $|t| < \tau_0$, which satisfies

$$\|u^t - u^0\|_X = \|g(t) - g(0)\|_X = \|DG(0)t + o(t)\|_X.$$

Consequently,

$$0 \le \frac{\|u^t - u^0\|_X}{t^{1/2}} \le \|Dg(0)\|_{\mathcal{L}(\mathbb{R},X)} t^{1/2} + \|\frac{o(t)}{t}\|_X t^{1/2} \to 0 \text{ for } t \to 0$$

and

$$\lim_{t\to 0} \frac{\|u^t - u^0\|_X}{t^{1/2}} = 0,$$

which also shows the convergence of u^t in the sense stated in (H2). ☐

Lemma 3.5. *Let U have a C^1-boundary, then the following assertions are true.*

1. *If $u \in L^p(U)$, then $t \mapsto u \circ F_t^{-1} \in C(\mathcal{T}, L^p(U))$ for all $1 \le p < \infty$.*
2. *If $u \in H^2(U)$, then $u \circ F_t^{-1} \in H^2(U)$. Moreover,*

$$\begin{aligned} \frac{d}{dt}(u \circ F_t^{-1})|_{t=0} &= -(Du)h \quad \text{exists in } H^1(U) \text{ and} \\ \frac{d}{dt}(D(u \circ F_t^{-1}))|_{t=0} &= -D((Du)h) \quad \text{exists in } L^2(U). \end{aligned}$$

Proof. Below, we will use several times that F_t^{-1} is a C^1-diffeomorphism on U for every fixed $t \in \mathcal{T}$ and also that $t \mapsto F_t^{-1} \in C(\mathcal{T}, C^1(\bar{U},\mathbb{R}^d))$.

1. As a result of Lemma 3.1, $u^p \in L^1(U)$ implies $u^p \circ F_t^{-1} \in L^1(U)$, hence $u \circ F_t^{-1} \in L^p(U)$. In order to prove the continuous dependence of $u \circ F_t^{-1}$ on t in $L^p(U)$, we show the following:

   $$\forall \epsilon > 0 \; \exists \delta > 0 \; \forall s,t \in \mathcal{T} \;\; |t-s| < \delta \;\Rightarrow\; \|u \circ F_t^{-1} - u \circ F_s^{-1}\|_{L^p(U)} < \epsilon.$$

   We know there exists a sequence $(u_m)_m \subset C(\bar{U})$ with $u_m \to u$ in $L^p(U)$ [Jos05]. Now, choose $\epsilon > 0$ arbitrarily and $m \in \mathbb{N}$ such that $\|u - u_m\|_{L^p(U)} < \epsilon$. Since u_m is uniformly continuous,

$|u_m(x) - u_m(y)| < \epsilon$ if $\|x-y\| < \bar{\delta}$ where $\bar{\delta} = \bar{\delta}(\epsilon) > 0$. In addition, $t \mapsto F_t{}^{-}1$ is uniformly continuous on the compact interval $\mathcal{T}$, hence, there exists a $\delta > 0$ such that $\|F_t{}^{-}1 - F_s^{-1}\|_{C(\bar{U},\mathbb{R}^d)} < \bar{\delta}$ if $|t-s| < \delta$ with $t, s \in \mathcal{T}$. Thus, using also Corollary 2.8 and Lemma 3.1, one deduces for $t, s \in \mathcal{T}$ with $|t-s| < \delta$ that

$$
\begin{aligned}
& \|u \circ F_t^{-1} - u \circ F_s^{-1}\|_{L^p(U)} \\
\leq\ & \|u \circ F_t^{-1} - u_m \circ F_t^{-1}\|_{L^p(U)} + \|u_m \circ F_t^{-1} - u_m \circ F_s^{-1}\|_{L^p(U)} + \|u_m \circ F_s^{-1} - u \circ F_s^{-1}\|_{L^p(U)} \\
=\ & \left(\int_\Omega |u-u_m|^p I_t dx\right)^{1/p} + \left(\int_\Omega |u_m(F_t^{-1}) - u_m(F_s^{-1})|^p dx\right)^{1/p} + \left(\int_\Omega |u-u_m|^p I_s dx\right)^{1/p} \\
\leq\ & \left(\beta \int_\Omega |u-u_m|^p dx\right)^{1/p} + \left(\int_\Omega \epsilon^p dx\right)^{1/p} + \left(\beta \int_\Omega |u-u_m|^p dx\right)^{1/p} \\
=\ & \beta^{1/p}\|u-u_m\|_{L^p(U)} + \epsilon|\Omega| + \beta^{1/p}\|u-u_m\|_{L^p(U)} \\
<\ & (2\beta^{1/p} + |\Omega|)\epsilon
\end{aligned}
$$

which proves the claim.

2. First, we show that for $f \in H^1(U)$ and $g \in C^{0,1}(U)$ the product satisfies $fg \in H^1(U)$. Since g is Lipschitz-continuous on U, which has also a C^1-boundary, we know that $g \in W^{1,\infty}(U)$ [Eva10]. Therefore,

$$
\begin{aligned}
& \int_\Omega (fg)^2 dx \leq \|f^2\|_{L^1(U)}\|g^2\|_{L^\infty(U)} < \infty, \\
& \int_\Omega \left((fg)'\right)^2 dx = \int_\Omega (f'g)^2 dx + 2\int_\Omega ff'gg' dx + \int_\Omega (fg')^2 dx \\
\leq\ & \|f'\|^2_{L^2(U)}\|g\|^2_{L^\infty(U)} + 2\|f\|_{L^2(U)}\|f'\|_{L^2(U)}\|g\|_{L^\infty(U)}\|g'\|_{L^\infty(U)} + \|f\|^2_{L^2(U)}\|g'\|^2_{L^\infty(U)} \\
<\ & \infty.
\end{aligned}
$$

Additionally, on can use a result from [KJF77] which states the following:

Let $G, O \subset \mathbb{R}^n$ be bounded and open and let $T : G \to O$ be surjective with

$$d\|x-y\| \leq \|T(x) - T(y)\| \leq c\|x-y\|$$

for all $x, y \in G$ with constants $c, d > 0$. Then

$$u \in H^1(O) \Rightarrow u \circ T \in H^1(G).$$

Since $F_t^{-1} : U \to U$ satisfies the above conditions, we deduce that for all $v \in H^1(U)$ the composition $v \circ F_t^{-1} \in H^1(U)$ is in the same space.

Now, let $u \in H^2(U)$, then $Du \in H^1(U, \mathbb{R}^d)$, hence $Du(F_t^{-1}) \in H^1(U, \mathbb{R}^d)$ and $(Du(F_t^{-1}))D(F_t{-}1) \in H^1(U, \mathbb{R}^d)$ due to the fact that $F_t^{-1} \in C^{1,1}(\bar{U}, \mathbb{R}^d)$ and, consequently, $D(F_t^{-1}) \in C^{0,1}(\bar{U}, \mathbb{R}^{d\times d})$. For sure, the result from [KJF77] can be applied to each component of vector- or matrix-valued function. All toghether, one concludes that

$$D(u \circ F_t^{-1}) = Du(F_t^{-1})D(F_t^{-1}) \in H^1(U)$$

and, therefore, $u \circ F_t^{-1} \in H^2(U)$.

In order to show the remaining two identities, we follow the proof of an even more general result given in [IKPb]. At the beginning, we show that

$$(u \circ F_t^{-1})(x) - u(x) = -t\int_0^1 Du(x + sth(x))h(x)ds \tag{12}$$

in $L^2(U)$ where $h \in C^{1,1}(\bar{U}, \mathbb{R}^d)$, $h|_{\partial U} = 0$ and $F_t = id + th$. Since $C^\infty(U)$ is dense in $H^2(U)$ with respect to the H^2-norm, there exists a $u_\epsilon \in C^\infty(U)$ with $\|u - u_\epsilon\|_{H^2(U)} < \epsilon$. Then one finds

$$\begin{aligned}
& \left\| u \circ F_t^{-1} - u + t \int_0^1 Du(\cdot + sth)hds \right\|_{L^2(U)} \leq \|u \circ F_t^{-1} - u_\epsilon \circ F_t^{-1}\|_{L^2(U)} + \|u_\epsilon - u\|_{L^2(U)} \\
+ & \left\| u_\epsilon \circ F_t^{-1} - u_\epsilon + t \int_0^1 Du_\epsilon(\cdot + sth)hds \right\|_{L^2(U)} \\
+ & \left\| t \int_0^1 \|Du_\epsilon(\cdot + sth) - Du(\cdot + sth)\| ds \right\|_{L^2(U)} \|h\|_{L^\infty(U)} \to 0
\end{aligned}$$

for $\epsilon \to 0$. This is true since the first two terms tend to zero due to the assumption and Lemma 3.1. The third term is zero anyway and the fourth term converges to zero due to the assumption and the boundedness of U. Using this expansion for the function u, we find

$$\left\| \frac{1}{t} \left(u \circ F_t^{-1} - u \right) + (Du)h \right\|_{L^2(U)}^2 \leq \int_\Omega \int_0^1 |Du\left(x + sth(x)\right) - Du(x)|^2 |h(x)|^2 dsdx \to 0$$

for $t \to 0$ using the first part of this lemma and the dominated convergence theorem together with the boundedness of U. Hence, $t \mapsto u \circ F_t^{-1}$ is differentiable at $t = 0$ in $L^2(U)$ with derivative $-(Du)h$. In order to show that $-(Du)h$ is also the derivative with respect to the H^1-topology, we have to show the corresponding convergence also for the weak derivatives in the $L^2(U)$-norm. For this case, we calculate the weak derivative of the right- (and hence left-) hand-side of equation (12), where we use integration by parts and the theorem of Fubini. Let $\varphi \in C_0^\infty(U)$, then

$$\begin{aligned}
& \left\langle \frac{\partial}{\partial x_i} \int_0^1 Du(\cdot + sth)hds, \varphi \right\rangle \\
= & -\int_\Omega \int_0^1 Du(x + sth(x))h(x)ds \frac{\partial}{\partial x_i} \varphi(x)dx = -\int_0^1 \int_\Omega Du(x + sth(x))h(x) \frac{\partial}{\partial x_i} \varphi(x)dxds = \\
= & \int_0^1 \int_\Omega \left(\sum_{k=1}^d \sum_{j=1}^d \frac{\partial^2}{\partial x_k \partial x_j} u(x + sth(x)) \left(\delta_{ij} + st \frac{\partial}{\partial x_i} h_j(x) \right) h_k(x) \right) \\
+ & \left(\sum_{k=1}^d \frac{\partial}{\partial x_k} u(x + sth(x)) \frac{\partial}{\partial x_i} h_k(x) \right) \varphi(x)dxds = \\
= & \int_0^1 \int_\Omega \left(h(x)^T D^2 u(x + sth(x))(I + stDh(x))_i + Du(x + sth(x))(Dh(x))_i \right) \varphi(x)dxds \\
= & \int_\Omega \int_0^1 \left(h(x)^T D^2 u(x + sth(x))(I + stDh(x))_i + Du(x + sth(x))(Dh(x))_i \right) ds\varphi(x)dx \\
= & \left\langle \int_0^1 \left(h^T D^2 u(\cdot + sth)(I + stDh)_i + Du(\cdot + sth)(Dh)_i \right) ds, \varphi \right\rangle.
\end{aligned}$$

Using this weak derivative, we deduce analogously to above that

$$\begin{aligned}
& \left\| D \left(\frac{1}{t}(u \circ F_t^{-1} - u) + (Du)h \right) \right\|_{L^2(U)}^2 \\
\leq & \int_\Omega \int_0^1 \| -h(x)^T D^2 u(x + sth(x))(I + stDh(x)) - Du(x + sth(x))Dh(x) \\
+ & h(x)^T D^2 u(x) + Du(x)Dh(x)\|^2 dsdx \\
\leq & \int_\Omega \int_0^1 \|h(x)^T \left(D^2 u(x + sth(x))(I + stDh(x)) - D^2 u(x) \right) \|^2 dsdx \\
+ & \int_\Omega \int_0^1 \| \left(Du(x + sth(x)) - Du(x) \right) Dh(x)\|^2 dsdx \to 0
\end{aligned}$$

for $t \to 0$ again using the first part of the lemma and the dominated convergence theorem.

The last assertion stated in the lemma is an immediate consequence of the calculations above, since

$$\left\|\left(\frac{1}{t}(D(u\circ F_t^{-1}) - Du) + D((Du)h)\right)\right\|^2_{L^2(U)} = \left\|D\left(\frac{1}{t}(u\circ F_t^{-1} - u) + (Du)h\right)\right\|^2_{L^2(U)} \to 0$$

which shows that $-D((Du)h)$ is the desired derivative with respect to the L^2-topology. □

Lemma 3.6. *1. Let $f \in C(\mathcal{T}, W^{1,1}(U))$ and assume that $f_t(0)$ exists in $L^1(U)$, then*

$$\frac{d}{dt}\int_{\Omega_t} f(t,x)dx_t|_{t=0} = \int_\Omega f_t(0,x)dx + \int_\Gamma f(0,x)h\cdot n d\Gamma.$$

2. Let $f \in C(\mathcal{T}, W^{2,1}(U))$ and assume that $f_t(0)$ exists in $W^{1,1}(U)$, then

$$\frac{d}{dt}\int_{\Gamma_t} f(t,x)d\Gamma_t|_{t=0} = \int_\Gamma f_t(0,x)d\Gamma + \int_\Gamma \left(\frac{\partial}{\partial n}f(0,x) + \kappa f(0,x)\right) h\cdot n d\Gamma,$$

where κ denotes the additive curvature of Γ, i.e. the sum of the $d-1$ principal curvatures of Γ.

Proof. Below we will present the main steps of the proof given in [Pei06].

1. One starts with inserting some useful terms in the following difference quotient which results in

$$\begin{aligned}
&\frac{1}{t}\left(\int_{\Omega_t} f(t,x)dx_t - \int_\Omega f(0,x)dx\right) = \frac{1}{t}\int_\Omega (I_t f(t,F_t(x)) - f(0,x))dx \\
=&\ \int_\Omega \frac{I_t-1}{t} f(t,F_t(x))dx + \frac{1}{t}\int_\Omega (f(t,F_t(x)) - f(t,x))dx \\
+&\ \frac{1}{t}\int_\Omega f(t,x) - f(0,x)dx =: X_t + Y_t + Z_t
\end{aligned}$$

Now, we show that

$$\lim_{t\to 0} X_t = \int_\Omega f(0,x)\operatorname{div} h\ dx$$

via considering that the difference

$$\begin{aligned}
&\left|X_t - \int_\Omega f(0,x)\operatorname{div} h\ dx\right| \\
\leq&\ \left|\int_\Omega \left(\frac{I_t-1}{t} - \operatorname{div} h)f(t,F_t(x)\right) dx\right| + \left|\int_\Omega (f(t,F_t(x)) - f(0,x))\operatorname{div} h\ dx\right| \\
\leq&\ \max_{x\in\bar\Omega}\left|\frac{I_t(x)-1}{t} - \operatorname{div} h(x)\right| \int_\Omega |f(t,F_t(x))|dx + \max_{x\in\bar\Omega}|\operatorname{div} h(x)| \int_\Omega |f(t,F_t(x)) - f(0,x)|dx \to 0
\end{aligned}$$

for $t\to 0$ due to Proposition 2.6 and the fact that $t\mapsto f(t,F_t(\cdot)) = f(t)\circ F_t \in C(\mathcal{T}, L^1(U))$, for details see [Pei06]. We next show that

$$\lim_{t\to 0} Y_t = \int_\Omega Df(0,x)h(x)dx.$$

For this case we apply the integral-mean-value-theorem to find

$$\begin{aligned}
&\left|Y_t - \int_\Omega Df(0,x)h(x)\ dx\right| \\
= &\left|\int_\Omega \left(\int_0^1 Df(t, x + s(F_t(x) - x))\frac{F_t(x) - x}{t} ds - Df(0,x)h(x)\right) dx\right| \\
\leq &\left|\int_\Omega \int_0^1 Df(t, x + s(F_t(x) - x))ds \left(\frac{F_t(x) - x}{t} - h(x)\right) dx\right| \\
+ &\left|\int_\Omega \int_0^1 \left(Df(t, x + s(F_t(x) - x)) - Df(0,x)\right) dsh(x)dx\right| \\
\leq &\max_{x\in\bar\Omega} \left\|\frac{F_t(x) - x}{t} - h(x)\right\| \int_0^1 \int_\Omega \|Df(t, x + s(F_t(x) - x))\| dxds \\
+ &\max_{x\in\bar\Omega} \|h(x)\| \int_0^1 \int_\Omega \|Df(t, x + s(F_t(x) - x)) - Df(0,x)\| dsdx \to 0
\end{aligned}$$

for $t \to 0$ due to Proposition 2.6 and an additional result stating that $t \mapsto Df(t, x + s(F_t(x) - x)) \in C(\mathcal{T}, L^1(U))$ for all $s \in [0,1]$ and that $\{Df(t, x + s(F_t(x) - x)) \mid s \in [0,1]\}$ is uniformly equicontinuous. For further details see again [Pei06]. Finally, we have

$$\lim_{t\to 0} Z_t = \int_\Omega f_t(0,x)dx$$

as a direct consequence of the existence of $f_t(0)$ in $L^1(U)$, which is assumed to hold. Now, we observe that

$$\int_\Omega f(0,x) \operatorname{div} h\ dx + \int_\Omega Df(0,x)h(x)dx = \int_\Omega \operatorname{div}(f(0,x)h)dx = \int_\Gamma f(0,x)h \cdot nd\Gamma$$

where we used the Stokes formula [DZ11], which is applicable since Γ is of class $C^{1,1}$ and thus, in particular, Lipschitzean. Combining the results obtained above yields the desired transformation formula.

2. Similar to the first part, we rearrange the difference quotient which gives

$$\begin{aligned}
&\frac{1}{t}\left(\int_{\Gamma_t} f(t,x)d\Gamma_t - \int_\Gamma f(0,x)d\Gamma\right) = \frac{1}{t}\int_\Gamma (w_t f(t, F_t(x)) - f(0,x))d\Gamma \\
= &\int_\Gamma \frac{w_t - 1}{t} f(t, F_t(x))d\Gamma + \frac{1}{t}\int_\Gamma (f(t, F_t(x)) - f(t,x))d\Gamma \\
+ &\frac{1}{t}\int_\Gamma (f(t,x) - f(0,x))d\Gamma =: X_t + Y_t + Z_t
\end{aligned}$$

The first term results in

$$\lim_{t\to 0} X_t = \int_\Gamma f(0,x) \operatorname{div}_\Gamma h\ d\Gamma$$

as a consequence of

$$\begin{aligned}
&\left|X_t - \int_\Gamma f(0,x) \operatorname{div}_\Gamma h\ d\Gamma\right| \\
\leq &\left|\int_\Gamma \left(\frac{w_t - 1}{t} - \operatorname{div}_\Gamma h) f(t, F_t(x)\right) d\Gamma\right| + \left|\int_\Gamma (f(t, F_t(x)) - f(0,x)) \operatorname{div}_\Gamma h\ d\Gamma\right| \\
\leq &\max_{x\in\Gamma} \left|\frac{w_t(x) - 1}{t} - \operatorname{div}_\Gamma h(x)\right| \int_\Gamma |f(t, F_t(x))| d\Gamma + \max_{x\in\Gamma} |\operatorname{div}_\Gamma h(x)| \int_\Gamma |f(t, F_t(x)) - f(0,x)| d\Gamma
\end{aligned}$$

Using the fact that $t \mapsto f(t, F_t(\cdot)) = f(t) \circ F_t \in C(\mathcal{T}, W^{1,1}(U))$ [Pei06], the trace theorem yields that the mapping

$$t \mapsto f(t, F_t(\cdot)) = f(t) \circ F_t \in C(\mathcal{T}, L^1(\Gamma)).$$

Consequently,

$$\left|X_t - \int_\Gamma f(0,x)\,\mathrm{div}_\Gamma\, h \; d\Gamma\right| \to 0$$

since $\|f(t,F_t(x)) - f(0,x)\|_{L^1(\Gamma)} \to 0$ and $\|f(t,F_t(x))\|_{L^1(\Gamma)}$ is bounded for $t \to 0$. Next, we prove

$$\lim_{t\to 0} Y_t = \int_\Gamma Df(0,x)h(x)d\Gamma.$$

Therefore, consider the estimate

$$\begin{aligned}
& \left|Y_t - \int_\Gamma Df(0,x)h(x)\; d\Gamma\right| \\
= & \left|\int_\Gamma \left(\int_0^1 Df(t,x+s(F_t(x)-x))\frac{F_t(x)-x}{t}ds - Df(0,x)h(x)\right) d\Gamma\right| \\
\leq & \left|\int_\Gamma \int_0^1 Df(t,x+s(F_t(x)-x))ds \left(\frac{F_t(x)-x}{t} - h(x)\right) d\Gamma\right| \\
+ & \left|\int_\Gamma \int_0^1 \left(Df(t,x+s(F_t(x)-x)) - Df(0,x)\right) ds h(x) d\Gamma\right| \\
\leq & \max_{x\in\Gamma}\left\|\frac{F_t(x)-x}{t} - h(x)\right\| \int_0^1 \int_\Omega \|Df(t,x+s(F_t(x)-x))\| d\Gamma ds \\
+ & \max_{x\in\Gamma}\|h(x)\| \int_0^1 \int_\Gamma \|Df(t,x+s(F_t(x)-x)) - Df(0,x)\| ds d\Gamma.
\end{aligned}$$

Applying once more that $t \mapsto Df(t,x+s(F_t(x)-x)) \in C(\mathcal{T}, W^{1,1}(U))$ for all $s \in [0,1]$ and that $\{Df(t,x+s(F_t(x)-x)) \mid s \in [0,1]\}$ is uniformly equicontinuous [Pei06], the trace theorem results in

$$\left|Y_t - \int_\Gamma Df(0,x)h(x)\; d\Gamma\right| \to 0$$

since $\|Df(t,x+s(F_t(x)-x)) - Df(0,x)\|_{L^1(\Gamma)} \to 0$ and $\|Df(t,x+s(F_t(x)-x))\|_{L^1(\Gamma)}$ is bounded for $t \to 0$. As a consequence of the assumed existence of $f_t(0)$ in $W^{1,1}(U)$, we know that $f_t(0)$ exists in $L^1(\Gamma)$ due to the trace theorem and, therefore,

$$\lim_{t\to 0} Z_t = \int_\Gamma f_t(0,x)d\Gamma$$

So far, we have

$$\frac{d}{dt}\int_{\Gamma_t} f(t,x)d\Gamma_t|_{t=0} = \int_\Gamma (f(0,x)\,\mathrm{div}_\Gamma\, h(x) + Df(0,x)h(x))d\Gamma + \int_\Gamma f_t(0,x)d\Gamma.$$

Moreover, we know that the tangential derivative, respectively divergence, is defined as

$$D_\Gamma f = (Df)_\Gamma - \frac{\partial f}{\partial n}n, \qquad \mathrm{div}_\Gamma\, h = (\mathrm{div}\, h)_\Gamma - (Dh)n\cdot n$$

and that the Green formula [DZ11] holds,

$$\int_\Gamma (f\,\mathrm{div}_\Gamma\, h + (D_\Gamma f)h)d\Gamma = \int_\Gamma f\kappa h\cdot n d\Gamma.$$

Thus,

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma_t} f(t,x)d\Gamma_t|_{t=0} &= \int_\Gamma (f(0,x)\,\mathrm{div}_\Gamma\, h(x) + D_\Gamma f(0,x)h(x) + \frac{\partial f}{\partial n}h\cdot n)d\Gamma + \int_\Gamma f_t(0,x)d\Gamma \\
&= \int_\Gamma (f\kappa h\cdot n + \frac{\partial f}{\partial n}h\cdot n)d\Gamma + \int_\Gamma f_t(0,x)d\Gamma \\
&= \int_\Gamma f_t(0,x)d\Gamma + \int_\Gamma \left(\frac{\partial}{\partial n}f(0,x) + \kappa f(0,x)\right) h\cdot n d\Gamma
\end{aligned}$$

which proves the claim. □

4 An Example

Let us investigate the following problem. Given a bounded $C^{1,1}$-domain $D \subset \mathbb{R}^d$, we consider a certain heat-distribution u_0 defined on $\mathbb{R}^d$ which we take as the initial value for the heat-equation at the time $t = 0$. The heat-distribution $u(t, x)$ with $t \geq 0$ and $x \in \mathbb{R}^d$ can thus be directly calculated using u_0 and the heat-kernel. The task is now to determine the "total heat" contained in D at the time $t = 1$ and to find a representation for its dependence on the domain D.

In the framework of the last sections, we therefore have $\Omega = D$ and $\Gamma = \partial\Omega$; besides, $U \subset \mathbb{R}^d$ may be any bounded domain satisfying $\bar{D} \subset U$. Moreover, we choose $X = L^2(\Omega)$, although the solutions to the heat-equation are of class $C^\infty(\mathbb{R}^d)$ for any fixed $t > 0$. For our situation we define

$$u_0(x) := \begin{cases} 1 & \text{if } \ x \in D \\ 0 & \text{otherwise} \end{cases}$$

as the initial heat-distribution. Furthermore, the shape functional is given by

$$J(u, \Omega) := \int_\Omega u \, dx$$

subject to

$$E(u, \Omega) := u - \frac{1}{2\sqrt{\pi}} \int_\Omega e^{-\frac{(\cdot - \xi)^2}{4}} d\xi = 0.$$

In this case, $E(u, \Omega)$ represents an explicit formula for the state u but, nevertheless, one can treat this constraint in the same way as we did in the last sections. In order to obtain an explicit representation of $\tilde{E}(u^t, t)$, we do the following transformations for any $\psi \in X_t := L^2(\Omega_t)$.

$$\begin{aligned}
\langle E(u_t, \Omega_t), \psi_t \rangle_{X_t^* \times X_t} &= \int_{\Omega_t} u_t(x_t)\psi_t(x_t) dx_t - \frac{1}{2\sqrt{\pi}} \int_{\Omega_t} \int_{\Omega_t} e^{-\frac{(\xi_t - x_t)^2}{4}} \psi_t(x_t) d\xi_t dx_t \\
= \quad & \int_{\Omega_t} u_t(x_t)\psi_t(x_t) dx_t - \frac{1}{2\sqrt{\pi}} \int_{\Omega_t} \int_\Omega e^{-\frac{(F_t(\xi) - x_t)^2}{4}} \psi_t(x_t) I_t(\xi) d\xi dx_t \\
= \quad & \int_\Omega u_t(F_t(x))\psi_t(F_t(x)) I_t(x) dx - \frac{1}{2\sqrt{\pi}} \int_\Omega \int_\Omega e^{-\frac{(F_t(\xi) - F_t(x))^2}{4}} \psi_t(F_t(x)) I_t(\xi) I_t(x) d\xi dx \\
= \quad & \int_\Omega u^t(x)\psi^t(x) I_t(x) dx - \frac{1}{2\sqrt{\pi}} \int_\Omega \int_\Omega e^{-\frac{(F_t(\xi) - F_t(x))^2}{4}} \psi^t(x) I_t(\xi) I_t(x) d\xi dx \\
=: \quad & \langle \tilde{E}(u^t, t), \psi^t \rangle_{X^* \times X}.
\end{aligned}$$

We further have to ensure that the assumptions (H1) - (H5) from the axiomatic description hold.

For (H1) observe that X is a Hilbert space and $\tilde{E}(u^t, t)$ is C^1 in both arguments due to the C^1-smoothness of F_t and I_t in the parameter t. In addition, $E(u_t, \Omega_t) = 0$ is equivalent to $\tilde{E}(u^t, t) = 0$ and $\tilde{E}(u, 0) = E(u, \Omega)$ for all $u \in X$ are direct consequences of the definition of E and $\tilde{E}$.

Per construction, $E(u, \Omega) = 0$ has trivially a unique solution u; moreover, (H6) is satisfied since

$$E_u(u, \Omega)\delta u = (\delta u, \cdot)_\Omega$$

implies that $\langle E_u(u, \Omega)\delta u, \psi \rangle_{X^* \times X} = \langle f, \psi \rangle_{X^* \times X}$ with $\psi \in X$ has a unique solution $\delta u \in X$ for every $f \in X^*$. Consequently, (H2) holds due to Proposition 3.4.

The assumption (H3) is trivially satisfied since $E(v, \Omega) - E(u, \Omega) - E_u(u, \Omega)(v - u) = 0$ in X^*.

Let $\psi \in X$, then

$$\begin{aligned}
& \left| \frac{1}{t} \langle (\tilde{E}(u^t, t) - \tilde{E}(u, t)) - (E(u^t, \Omega) - E(u, \Omega)), \psi \rangle_{X^* \times X} \right| \\
= \quad & \left| \frac{1}{t} \left(\int_\Omega (u^t - u)\psi I_t dx - \int_\Omega (u^t - u)\psi dx \right) \right| = \left| \int_\Omega \frac{u^t - u}{t} \psi (I_t - 1) dx \right| \\
\leq \quad & \left\| \frac{u^t - u}{t^{1/2}} \right\|_{L^2(\Omega)} \left\| \frac{I_t - 1}{t^{1/2}} \right\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)} \to 0
\end{aligned}$$

due to (H2) and Proposition 2.6. Hence, (H4) holds.

Finally, (H5) is true since $j_1 = id \in C^{1,1}(\mathbb{R},\mathbb{R})$. In addition, the adjoint equation

$$\langle E_u(u,\Omega)\psi, p\rangle_{X^*\times X} = (j_1'(u),\psi)_\Omega, \quad \psi \in X$$

which explicitly reads

$$\int_\Omega \psi p dx = \int_\Omega \psi dx, \quad \psi \in X$$

admits the unique solution $p(x) = 1$ with $x \in \Omega$ since $E_u(u,\Omega)\delta u = (\delta u, \cdot)_\Omega$ and $j_1' = id$.

Therefore, all assumptions made in Theorem 3.2 are satisfied and we know that the shape derivative of J at the "point" Ω exists in every "direction" $h \in C^{1,1}(\bar{U},\mathbb{R}^d)$. In order to apply Theorem 3.2 , let us first do the following calculation, where we consider the solution u and the adjoint state p as elements of $L^2(U)$. As u is a C^∞-function on U and p is constant on Ω, there are no difficulties with such an extension.

$$\begin{aligned}
&\frac{d}{dt}\langle \tilde{E}(u,t), p\rangle_{X^*\times X}|_{t=0} \\
=\ & \frac{d}{dt}\left(\int_{\Omega_t} u(F_t^{-1}(x_t))p(F_t^{-1}(x_t))dx_t - \frac{1}{2\sqrt{\pi}}\int_{\Omega_t}\int_{\Omega_t} e^{-\frac{(\xi_t-x_t)^2}{4}} p(F_t^{-1}(x_t))d\xi_t dx_t\right)|_{t=0} \\
=\ & \frac{d}{dt}\left(\int_{\Omega_t} u(F_t^{-1}(x_t))dx_t - \frac{1}{2\sqrt{\pi}}\int_{\Omega_t}\int_{\Omega_t} e^{-\frac{(\xi_t-x_t)^2}{4}} d\xi_t dx_t\right)|_{t=0} \\
=\ & \int_\Omega Du(-h)dx + \int_\Gamma uh\cdot nd\Gamma \\
-\ & \frac{1}{2\sqrt{\pi}}\left(\int_\Omega\left(\frac{d}{dt}\int_{\Omega_t} e^{-\frac{(\xi_t-x_t)^2}{4}} d\xi_t|_{t=0}\right)dx + \int_\Gamma\left(\int_\Omega e^{-\frac{(\xi-x)^2}{4}}d\xi\right)h\cdot nd\Gamma\right) \\
=\ & \int_\Omega Du(-h)dx + \int_\Gamma uh\cdot nd\Gamma \\
-\ & \frac{1}{2\sqrt{\pi}}\left(\int_\Omega\left(\int_\Omega 0d\xi + \int_\Gamma e^{-\frac{(\xi-x)^2}{4}}h\cdot nd\Gamma\right)dx + \int_\Gamma\left(\int_\Omega e^{-\frac{(\xi-x)^2}{4}}d\xi\right)h\cdot nd\Gamma\right) \\
=\ & \int_\Omega Du(-h)dx + \int_\Gamma uh\cdot nd\Gamma - \frac{1}{2\sqrt{\pi}}\left(\int_\Omega\int_\Gamma e^{-\frac{(\xi-x)^2}{4}}h\cdot nd\Gamma dx + \int_\Gamma\int_\Omega e^{-\frac{(\xi-x)^2}{4}}d\xi h\cdot nd\Gamma\right) \\
=\ & \int_\Omega Du(-h)dx + \int_\Gamma uh\cdot nd\Gamma - \int_\Gamma uh\cdot nd\Gamma - \int_\Gamma uh\cdot nd\Gamma \\
=\ & \int_\Omega Du(-h)dx - \int_\Gamma uh\cdot nd\Gamma.
\end{aligned}$$

Here we used Lemma 3.6 and the theorem of Fubini, which is applicable due to the smoothness and boundedness of Ω and the integrability of the objective function. In detail, one uses the definition of the boundary integral and concludes for each local $C^{1,1}$-diffeomorphism $\varphi : S \to U$ with $S \subset \mathbb{R}^{d-1}$ open and $U \subset \Gamma$ open that

$$\begin{aligned}
& \int_\Omega\int_\Gamma e^{-\frac{(\xi-x)^2}{4}}h\cdot nd\Gamma dx = \int_\Omega\int_S e^{-\frac{(\varphi(s)-x)^2}{4}}h\cdot n(\det(D\varphi^T D\varphi))^{1/2}dsdx \\
=\ & \int_S\int_\Omega e^{-\frac{(\varphi(s)-x)^2}{4}}h\cdot ndx(\det(D\varphi^T D\varphi))^{1/2}ds = \int_\Gamma\int_\Omega e^{-\frac{(\xi-x)^2}{4}}h\cdot ndxd\Gamma.
\end{aligned}$$

Finally, we find the representation for the Eulerian derivative, which is given by

$$\begin{aligned}
dJ(u,\Omega)h &= -\frac{d}{dt}\langle \tilde{E}(u,t), p\rangle_{X^*\times X}|_{t=0} + \int_\Omega j_1(u)\operatorname{div} h dx \\
&= \int_\Omega ((Du)h + u\operatorname{div} h)\, dx + \int_\Gamma uh\cdot nd\Gamma = \int_\Omega (\operatorname{div}(uh))\, dx + \int_\Gamma uh\cdot nd\Gamma \\
&= 2\int_\Gamma uh\cdot nd\Gamma.
\end{aligned}$$

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